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Symmetries, conservation laws and multisoliton perturbation theory*

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Abstract. On the basis of action/angle variables for multisolitons, new symmetries (mastersymmetries) are constructed. For systems with known hierarchies of non-Hamiltonian mastersymmetries, hierarchies of Hamiltonian mastersymmetries are constructed and for systems with known Hamiltonian mastersymmetries, a hierarchy of non-Hamiltonian mastersymmetries are constructed. Moreover, with the help of the action/angle variables, the N-soliton perturbation theory, on the soliton submanifold (adiabatic approximation), is formulated. An explicit form of the time evolution of asymptotic data under the influence of perturbation is presented.

Introduction

In the previous paper [1], hereafter denoted as I, we presented the construction of action/angle variables for multisoliton systems. It turns out that the action and angle variables on an N-soliton manifold can be obtained directly via partial derivatives (with respect to the asymptotic data) of a fundamental scalar field. Moreover, the suitable action/angle vector fields are expressible by the same partial derivatives of a field variable and are in close connection with eigenstates of a recursion operator.

In the present paper we consider other subalgebras of soliton symmetries. It is well known from the literature that for soliton systems two kinds of algebra of symmetries exist. The first algebra is connected with the systems for which there exists the so-called recursion operator in an explicit form (the Korteweg-de Vries equation for example). Such an algebra (called hereditary algebra [2]) consists of time-independent Hamiltonian symmetries and linear-in-time non-Hamiltonian symmetries (represented by the so-called non-Hamiltonian mastersymmetries [3]). The second algebra is connected with the systems for which there are no recursion operators in the explicit form (the Benjamin-Ono equation for example). Such an algebra (called non-canonical action/angle algebra [4]) consists again of time-independent Hamiltonian symmetries and linear-in-time Hamiltonian symmetries (represented by the so-called Hamiltonian mastersymmetries [5]).

So, a question arises of whether it is possible to construct a hierarchy of Hamiltonian mastersymmetries for the systems with known hereditary algebra and a hierarchy of non-Hamiltonian mastersymmetries for the systems with known non-canonical action/angle algebra. The affirmative answer (on the soliton manifold at least) is presented in section 3, where the explicit construction of missing matersymmetries is

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given. In section 4, we construct the multisoliton perturbation theory, in a purely algebraic way, on the basis of non-canonical action/angle scalar fields. As we confine ourselves to the soliton manifold only, the perturbation theory is given in the so-called *adiabatic approximation. We finish this paper with a few examples illustrating the* results.

1. Basic facts

We consider evolution equations

$$u_t = K_1(u) \tag{1.1}$$

on a suitable manifold M. We are interested in equations for which there exists a hierarchy of vector fields $K_n(u)$ (symmetries), a scaling vector field $S(u)$ and the Poisson (implectic) operator θ_0 so that $K_n(u)$ and $S(u)$ are Hamiltonian vector fields with respect to θ_0:

$$K_n(u) = \theta_0 \operatorname{grad} H_n(u) \qquad S(u) = \theta_0 \operatorname{grad} F(u) \tag{1.2}$$

where $H_n(u)$ and $F(u)$ are suitable scalar fields on the manifold under consideration, and they fulfil the following commutation relations in the Lie algebra of vector fields:

$$L_{K_n} K_m = [K_n, K_m] = 0 \qquad L_S K_m = [S, K_m] = (m+\alpha) K_m \qquad \alpha = \text{const.} \tag{1.3}$$

In this paper we confine our considerations to the N-soliton submanifold M_N of M and to such systems (1.1)–(1.3) whose N-soliton solutions decompose asymptotically for $t \to \infty$ into the one-soliton form

$$u_N \simeq \sum_{i=1}^{N} s_i(x + c_i t + q_i). \tag{1.4}$$

If the speeds c_i and the phases q_i are considered as variables then the set of these solutions forms a $2N$-dimensional invariant submanifold M_N of M.

For the submanifold M_N we may provide a new parametrization in the following way. We define a map Π which assigns to each $u_N(x, t)$ the set of asymptotic data $\bar{u} = (q_1, \ldots, q_N, c_1, \ldots, c_N)^{\mathrm{T}}$. It can be observed that although we refer to the asymptotic form of the N-solitons, this new parametrization is defined for arbitrary time.

Lemma 1. The quantities q_i, c_i are scalar fields on the submanifold M_N with the following time dependence:

$$\frac{\partial}{\partial t} q_i(t) = c_i \qquad \frac{\partial}{\partial t} c_i(t) = 0. \tag{1.5}$$

The proof is given in I.

Lemma 1 shows that the flow (1.1) is linearized in our new coordinates. For convenience, we shall call the manifold M_N parametrized in (x, t) coordinates the physical space (nonlinear space), and the same manifold endowed with the coordinates (q_i, c_i) the linear space.

Now the general procedure to recover the algebraic structure of dynamical system (1.1) on M_N is very simple in principle: first find the structure of the linear system (1.5), then carry over the whole structure to the system represented in physical coordinates. Although we don't know the explicit form of our variable transformation Π, we know how tensor fields behave under a change of coordinates. Actually Π induces the pushforward

$$\Pi' : T_u M_N \to T_{\bar{u}} \mathbb{R}^{2N} \tag{1.6}$$

which maps vector fields of the nonlinear space onto vector fields on the linear space, and is a Lie algebra isomorphism, and the pullback

$$\Pi^+ = (\Pi')^* : T^*_{\bar{u}} \mathbb{R}^{2N} \to T^*_u M_N \tag{1.7}$$

which is the transpose of Π' WRT the duality between the tangent and the cotangent bundle and maps covector fields of the linear space onto covector fields on the nonlinear space.

2. Structure of a linear system

In this section we restrict our considerations to the linear system (1.5) and examine its algebraic structure. First, one should observe that the system (1.5) may be given in Hamiltonian form. Let us recall that an equation is Hamiltonian if the flow is of the form θ grad H, where θ is an impletic (Poisson) operator. Every Poisson operator θ defines a natural Poisson bracket of scalar fields f and g in the following way:

$$\{f, g\}_\theta = \langle \text{grad } g, \theta \text{ grad } f \rangle. \tag{2.1}$$

Equation (1.5) admits many different Hamiltonian formulations. For every $p \neq 2$

$$\bar{u}_t = \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda_p \\ -\Lambda_p & 0 \end{pmatrix} \text{grad} \left(\frac{1}{2-p} \sum_{i=1}^{N} c_i^{2-p} \right) \tag{2.2}$$

holds. Here Λ_p denotes the diagonal $N \times N$ matrix

$$\Lambda_p = \begin{pmatrix} c_1^p & & 0 \\ & \ddots & \\ 0 & & c_N^p \end{pmatrix}. \tag{2.3}$$

To make the following compatible with formulae (1.2) and (1.3) we choose the Poisson operator given by

$$\bar{\theta}_0 = \begin{pmatrix} 0 & \Lambda_{1-\alpha} \\ -\Lambda_{1-\alpha} & 0 \end{pmatrix} \tag{2.4}$$

where α is the normalization factor introduced in (1.3).

Some useful results concerning the linear system (1.5) were obtained in I. Here we collect them in the following way:

Lemma 2.

(i) The operator

$$\bar{\phi}=\begin{pmatrix}\Lambda_1 & 0\\ 0 & \Lambda_1\end{pmatrix} \tag{2.5}$$

is the hereditary recursion operator which maps symmetries of equation (1.5) again onto symmetries [6].

(ii) The eigenvalues of $\bar{\phi}$ are $c_1, \ldots, c_N$, each of them occurs twice. The eigenvectors $\bar{A}_i$ and $\bar{B}_i$ of $\bar{\phi}$ WRT c_i are given by partial derivatives of the 'field variable' $\bar{u}=(q_1, \ldots, q_n, c_1, \ldots, c_N)^{\mathrm{T}}$ WRT the coordinates

$$\bar{A}_i=\frac{\partial \bar{u}}{\partial q_i} \qquad \bar{B}_i=\frac{\partial \bar{u}}{\partial c_i} \qquad \bar{\phi}\bar{A}_i(\bar{B}_i)=c_i\bar{A}_i(\bar{B}_i). \tag{2.6}$$

At each point the tangent space $T_{\bar{u}}\mathbb{R}^{2n}$ is the span of the eigenvectors of $\bar{\phi}$.

(iii) WRT the Poisson bracket given by $\bar{\theta}_0$ the coordinates c_i^{α}, q_i fulfil for all $i, j=1, \ldots, N$ the following relations:

$$\{c_i^{\alpha}, q_i\}_{\bar{\theta}_0}=\alpha\delta_{ij} \qquad \{c_i^{\alpha}, c_j^{\alpha}\}_{\bar{\theta}_0}=\{q_i, q_j\}_{\bar{\theta}_0}=0. \tag{2.7}$$

Hence $((1/\alpha)c_i^{\alpha}, q_i)$ are the canonical coordinates corresponding to $\bar{\theta}_0$. They are called canonical action/angle variables.

(iv) For every $i=1, \ldots, N$, canonical action/angle vector fields are of the form

$$\bar{A}_i=\bar{\theta}_0 \operatorname{grad}\left(\frac{1}{\alpha} c_i^{\alpha}\right) \tag{2.8a}$$

$$c_i^{1-\alpha}\bar{B}_i=\alpha\frac{\partial \bar{u}}{\partial c_i^{\alpha}}=\bar{\theta}_0 \operatorname{grad}(-q_i). \tag{2.8b}$$

(v) The symmetries $\bar{K}_n: \bar{\phi}^n(c_1, \ldots, c_N, 0, \ldots, 0)^{\mathrm{T}}$ are Hamiltonian vector fields with respect to the $\bar{\theta}_0$,

$$\bar{K}_n=(c_1^n, \ldots, c_N^n, 0, \ldots, 0)^{\mathrm{T}}=\bar{\theta}_0 \operatorname{grad}\left(\frac{1}{n+\alpha}\sum_{i=1}^{N} c_i^{n+\alpha}\right)=\bar{\theta}_0\nabla\bar{H}_n \tag{2.9}$$

and commute in pairs,

$$[\bar{K}_n, \bar{K}_m]=0. \tag{2.10}$$

They are also Hamiltonian vector fields with respect to the higher-order Poisson operators $\bar{\theta}_r=\bar{\phi}^r\bar{\theta}_0$.

(vi) The Hamiltonian vector field

$$\begin{aligned}\bar{S}&=(-\alpha q_1, \ldots, -\alpha q_N, c_1, \ldots, c_N)^{\mathrm{T}}\\ &=\bar{\theta}_0 \operatorname{grad}\left(-\sum_{i=1}^{N} c_i^{\alpha} q_i\right)=\bar{\theta}_0\nabla\bar{F}\end{aligned} \tag{2.11}$$

is a scaling vector field for $\bar{K}_n$, i.e.

$$[\bar{S}, \bar{K}_n]=(n+\alpha)\bar{K}_n. \tag{2.12}$$

Observe that the angle vector fields (2.8b) as well as the scaling one are Hamiltonian vector fields only WRT $\bar{\theta}_0$.

Now, for the sake of further considerations, including the recovery of full structure in the Lie algebra of the vector fields for the nonlinear soliton systems, let us define the fundamental algebra $\bar{\mathscr{A}}$ consisting of the following vector fields on the linear space:

$$\bar{P}_{r,n}=\bar{\phi}^n\sum_{i=1}^N q_i^r\bar{A}_i \qquad \bar{M}_{s,n}=\bar{\phi}^{n+1}\sum_{i=1}^N q_i^s\bar{B}_i. \tag{2.13a}$$

These vector fields fulfil the commutator relations

$$\begin{aligned}&[\bar{P}_{r,n},\bar{P}_{s,m}]=(s-r)\bar{P}_{r+s-1,n+m}\\&[\bar{M}_{r,n},\bar{M}_{s,m}]=(m-n)\bar{M}_{r+s,n+m}\\&[\bar{P}_{r,n},\bar{M}_{s,m}]=s\bar{M}_{s+r-1,n+m}-n\bar{P}_{s+r,n+m}.\end{aligned} \tag{2.13b}$$

Generally, the fundamental algebra (2.13) has a very rich structure and contains a lot of information on corresponding soliton systems. More details will be presented in a separate paper. Here we confine ourselves to two important subalgebras of $\bar{\mathscr{A}}$ which are very useful when applied to soliton theory.

Let us introduce three hierarchies of vector fields:

$$\begin{aligned}&\bar{K}_n=\bar{P}_{0,n}=\bar{\phi}^n\bar{K}_0 \qquad \bar{\tau}_n=\bar{M}_{0,n}-\alpha\bar{P}_{1,n}=\bar{\phi}^n\bar{\tau}_0\\&\bar{S}_n^p=\bar{M}_{0,n}-(n+\alpha-p)\bar{P}_{1,n}.\end{aligned} \tag{2.14}$$

Notice that $\bar{K}_n$ are simple symmetries (2.9) of our linear system and $\bar{\tau}_0=\bar{S}_0^0=\bar{S}$ is a scaling vector field (2.11).

By straightforward calculations we now obtain the following results:

Lemma 3.

(i) Vector fields $\bar{K}_n$ and $\bar{\tau}_n$ fulfil the commutator relations

$$[\bar{K}_n,\bar{K}_m]=0 \qquad [\bar{\tau}_n,\bar{K}_m]=(m+\alpha)\bar{K}_{n+m} \qquad [\bar{\tau}_n,\bar{\tau}_m]=(m-n)\bar{\tau}_{n+m}. \tag{2.15}$$

(ii) Vector fields $\bar{K}_n$ and $\bar{S}_n^p$ fulfil the commutator relations

$$\begin{aligned}&[\bar{K}_n,\bar{K}_m]=0 \qquad [\bar{S}_n^p,\bar{K}_m]=(n+m+\alpha-p)\bar{K}_{n+m}\\&[\bar{S}_n^p,\bar{S}_m^p]=(m-n)\bar{S}_{n+m}^p.\end{aligned} \tag{2.16}$$

Applying the above commutator relations and the definition of mastersymmetries [3, 5] we find that vector fields $\bar{\tau}_n$ and $\bar{S}_n^p$ are mastersymmetries of degree one for the linear system (1.5). $\bar{\tau}_n$ mastersymmetries are non-Hamiltonian vector fields and we call the set $(\bar{K}_n,\bar{\tau}_n)$ the hereditary algebra (or the weak action/angle algebra). In contrast, the symmetries $\bar{K}_n$ and mastersymmetries $\bar{S}_n^p$ are Hamiltonian vector fields with respect to the Poisson operator $\bar{\theta}_p$:

$$\bar{K}_n=\bar{\theta}_r\operatorname{grad}\bar{H}_{n-r} \qquad \bar{S}_n^p=\bar{\theta}_p\operatorname{grad}\bar{T}_{n-p} \tag{2.17a}$$

where

$$\bar{H}_n=\frac{1}{n+\alpha}\sum_{i=1}^N c_i^{n+\alpha} \qquad \bar{T}_n=-\sum_{i=1}^N q_ic_i^{n+\alpha}. \tag{2.17b}$$

Scalar fields $\bar{H}_n$ and $\bar{T}_n$ play the role of action/angle variables which are non-canonical WRT $\bar{\theta}_p$:

$$\begin{aligned}&\{\bar{H}_n,\bar{H}_m\}_{\bar{\theta}_p}=0 \qquad \{\bar{T}_n,\bar{H}_m\}_{\bar{\theta}_p}=(n+m+\alpha+p)\bar{H}_{n+m+p}\\&\{\bar{T}_n,\bar{T}_m\}_{\bar{\theta}_p}=(m-n)\bar{T}_{n+m+p}.\end{aligned} \tag{2.18}$$

Hence, for convenience, we call scalar fields $\bar{H}_n$ and $\bar{T}_n$ the non-canonical action/angle variables, and the set $(\bar{K}_n, \bar{S}_n^p)$ the non-canonical pth action/angle algebra.

3. Algebraic structure on the physical space

In the previous section we found the algebraic structure of the linear system (1.5). But of course, we would like to carry over the structure on the physical N-soliton manifold and express all desired quantitites in terms of the field variable $u_N(x, t)$. Partially this was done in I whose main results are collected in theorem 1:

Theorem 1.

(i) $\phi = (\Pi')^{-1}\bar{\phi}\Pi' \colon T_u M_N \to T_u M_N$ is a hereditary recursion operator on the physical space.

(ii) The eigenvectors

$$\begin{aligned} A_i &= \frac{\partial u_N}{\partial q_i} = \theta_0 \operatorname{grad}\left(\frac{1}{\alpha} E_i^\alpha\right) \\ c_i^{1-\alpha} B_i &= c_i^{1-\alpha} \frac{\partial u_N}{\partial c_i} = \theta_0 \operatorname{grad}(-\Omega_i) \end{aligned} \tag{3.1}$$

of $\phi(u_N)$ are Hamiltonian vector fields WRT the Poisson operator $\theta_0 = \theta|_{\rm red}$ determined by

$$S(u) = \theta(u) \operatorname{grad} F(u). \tag{3.2}$$

$\theta(u)$ is the Poisson operator from the whole manifold M and $\theta|_{\rm red}$ is its reduction to M_N. Moreover $\theta|_{\rm red} = \theta_0$ is related to $\bar{\theta}_0$ by

$$\theta_0(u_N) = (\Pi')^{-1}\bar{\theta}_0(\Pi^+)^{-1}. \tag{3.3}$$

(iii) The potentials E_i^α and Ω_i of the eigenvectors A_i and $c_i^{1-\alpha}B_i$ are given by the partial derivatives

$$E_i^\alpha = -\frac{\partial F(u_N)}{\partial q_i} \qquad \Omega_i = -\frac{\partial F(u_N)}{\partial c_i^\alpha}. \tag{3.4}$$

(iv) $(1/\alpha)E_i^\alpha$ and Ω_i are canonical coordinates WRT θ_0, i.e. for al $i, j = 1, \ldots, N$ the following holds:

$$\{E_i^\alpha, E_j^\alpha\}_{\theta_0} = \{\Omega_i, \Omega_j\}_{\theta_0} = 0 \qquad \{E_i^\alpha, \Omega_j\}_{\theta_0} = \alpha\delta_{ij}. \tag{3.5}$$

In this section we develop the idea of the so-called soliton fundamental algebra of vector fields. It allows us to recover in a common way the hierarchies of symmetries and mastersymmetries known for soliton systems. Furthermore, it enables us to construct the hierarchy of Hamiltonian mastersymmetries for the systems with a recursion operator and the hierarchy of non-Hamiltonian mastersymmetries for the soliton systems without a recursion operator in explicit form (in the non-extended sense).

Via the inverse of the pushforward Π' we define the soliton fundamental algebra $\mathscr{A}$ as the image of $\bar{\mathscr{A}}$ under $(\Pi')^{-1}$:

$$\mathscr{A} = (\Pi')^{-1}\bar{\mathscr{A}}.$$

Then the basic vector fields have the form

$$\begin{aligned} (\Pi')^{-1}\bar{P}_{r,n} &= P_{r,n}(u_N) = \phi^n \sum_{i=1}^N q_i^r A_i = \sum_{i=1}^N q_i^r c_i^n A_i \\ (\Pi')^{-1}\bar{M}_{s,n} &= M_{s,n}(u_N) = \phi^{n+1} \sum_{i=1}^N q_i^s B_i = \sum_{i=1}^N q_i^s c_i^{n+1} B_i. \end{aligned} \tag{3.6}$$

Since the pushforward is a Lie algebra isomorphism the commutator relations (2.13b) of $\bar{\mathscr{A}}$ are also valid for $\mathscr{A}$. The same holds for all subalgebras.

Hence, we found the representation of commuting symmetries $K_n(u_N)$, non-Hamiltonian mastersymmetries $\tau_n(u_N) = M_{0,n}(u_N) - \alpha P_{1,n}(u_N)$ and the Hamiltonian mastersymmetries $S_n^p(u_N) = M_{0,n}(u_N) - (n-p+\alpha)P_{1,n}(u_N)$.

It means that, at least on the N-soliton manifold M_N, each dynamical system (1.1)-(1.3) contains hereditary algebra (2.15) as well as non-canonical action/angle algebra (2.16). This statement is one of the important results of our paper. Since the non-canonical action/angle algebra is a new object for a hierarchy which admits a recursion operator in explicit form, we give a full structure of the algebra in the following lemma.

Lemma 4. On the N-soliton manifold the Hamiltonian vector fields

$$K_n(u_N) = \theta_p \operatorname{grad} H_{n-p} \qquad S_n^p(u_N) = \theta_p \operatorname{grad} T_{n-p} \tag{3.7}$$

have the following representation:

$$K_n(u_N) = \sum_{i=1}^N \phi^n A_i = \sum_{i=1}^N c_i^n A_i \tag{3.8a}$$

$$\begin{aligned} S_n^p(u_N) &= \sum_{i=1}^N \phi^{n+1} B_i - (n-p+\alpha) \sum_{i=1}^N \phi^n q_i A_i = \tau_n - (n-p) \sum_{i=1}^N \phi^n q_i A_i \\ &= \left(\frac{n-p}{\alpha} + 1\right) \tau_n - \frac{n-p}{\alpha} \sum_{i=1}^N \phi^{n+1} B_i. \end{aligned} \tag{3.8b}$$

The corresponding scalar fields are given by

$$H_n(u_N) = \frac{1}{n+\alpha} \sum_{i=1}^N c_i^n E_i^\alpha \qquad T_n(u_N) = -\sum_{i=1}^N c_i^{n+\alpha} \Omega_i. \tag{3.9}$$

Proof. Since the formulae for the vector fields are obvious by construction we only prove the representation of the scalar fields:

$$\begin{aligned} H_n(u_n) &= \frac{1}{n+\alpha} L_{\tau_0} H_n = -\frac{1}{n+\alpha} \langle \operatorname{grad} F, K_n \rangle \\ &= -\frac{1}{n+\alpha} \sum_{i=1}^N c_i^n \langle \operatorname{grad} F, A_i \rangle = \frac{1}{n+\alpha} \sum_{i=1}^N c_i^n E_i^\alpha \end{aligned} \tag{3.10a}$$

$$\begin{aligned} T_n(u_N) &= \frac{1}{n} L_{\tau_0} T_n = -\frac{1}{n} \langle \operatorname{grad} F, S_n^0 \rangle \\ &= \frac{1}{\alpha} \sum_{i=1}^N c_i^{n+1} \langle \operatorname{grad} F, B_i \rangle = -\sum_{i=1}^N c_i^{n+\alpha} \Omega_i. \end{aligned} \tag{3.10b}$$

At the end of this section we again turn our attention to the scalar fields on M_N. By construction we know that the Poisson manifold P_{θ_0} of scalar fields over the physical space and the corresponding one $P_{\bar{\theta}_0}$ of linear space are isomorphic. However, we are now able to give this isomorphism explicitly. We define a map $P: P_{\theta_0} \to P_{\bar{\theta}_0}$, which simply assigns to every scalar field

$$f(u_N) = \int_{-\infty}^{+\infty} \dots \mathrm{d}x \in P_{\theta_0} \tag{3.11}$$

the evaluation of the integral on M_N. For all our examples of section 5 as well as for other ones of I we find the following relations:

$$P(E_i^\alpha)=\beta c_i^\alpha \qquad P(\Omega_i)=\beta q_i \qquad P(H_n)=\beta\bar{H}_n \qquad P(T_n)=\beta\bar{T}_n \tag{3.12}$$

where β is some constant depending on the considered equation. Hence, the map P/β is the desired Lie algebra isomorphism. This map P was already introduced in [7] and was used to give an interpretation of interacting solitons as field representatives of Galilean point particles [4, 8].

4. Multisolution perturbation theory

For the dynamical systems (1.1), on the basis of the previous considerations, we are now able to formulate the multisoliton perturbation theory. It follows from the fact that we have at our disposal a complete set of conserved quantities for (1.1) on the N-soliton manifold as well as a suitable set of vector fields forming the basis of a tangent bundle to N-soliton flow.

We consider the evolution equation (1.1) with N-soliton solutions admitting asymptotic behaviour in the form (1.4). As we proved in I, (1.1) always possess a recursion operator ϕ on M_N in explicit or implicit form with the following eigenstates:

$$\begin{aligned} &A_i=\frac{\partial u_N}{\partial q_i} \qquad \phi A_i=c_iA_i \\ &B_i=\frac{\partial u_N}{\partial c_i} \qquad \phi B_i=c_iB_i \qquad A_i,\, B_i\in T_uM_N. \end{aligned} \tag{4.1}$$

Provided there is implectic–symplectic factorization of ϕ, i.e. $\phi=\theta_0 J$, we find

$$\begin{aligned} &A_i^*=JA_i=c_i\theta_0^{-1}A_i \qquad \phi^*A_i^*=c_iA_i^* \\ &B_i^*=JB_i=c_i\theta_0^{-1}B_i \qquad \phi^*B_i^*=c_iB_i^* \qquad A_i^*,\, B_i^*\in T_u^*M_N \end{aligned} \tag{4.2}$$

where $\phi^*=(\theta_0 J)^*=J\theta_0$ is the recursion operator for adjoint symmetries of the cotangent bundle $T_u^*M_N$.

As the complete set of conserved quantities we choose the non-canonical action/angle variables

$$H_n=\frac{1}{n+\alpha}\sum_{i=1}^{N}c_i^nE_i^\alpha=\frac{\beta}{n+\alpha}\sum_{i=1}^{N}c_i^{n+\alpha} \tag{4.3a}$$

$$T_n=-\sum_{i=1}^{N}c_i^{n+\alpha}\Omega_i=-\beta\sum_{i=1}^{N}c_i^{n+\alpha}q_i. \tag{4.3b}$$

The suitable gradients and vector fields are as follows:

$$\nabla H_n=\sum_{i=1}^{N}c_i^{n-1}A_i^* \tag{4.4a}$$

$$\nabla T_n=\sum_{i=1}^{N}[c_i^nB_i^*-(n+\alpha)c_i^{n-1}q_iA_i^*] \tag{4.4b}$$

$$K_n=\theta_0\nabla H_n=\sum_{i=1}^{N}c_i^nA_i \tag{4.5a}$$

$$S_n=S_n^0=\theta_0\nabla T_n=\sum_{i=1}^{N}[c_i^{n+1}B_i-(n+\alpha)c_i^nq_iA_i]. \tag{4.5b}$$

Now let us consider the perturbed mth equation from the hierarchy of (1.1):

$$u_t = K_m(u) + \varepsilon R(u) \tag{4.6}$$

where ε is the smallness parameter. In our considerations we confine ourselves to perturbation on M_N, so we perform all calculations in the so-called adiabatic approximation. What do we mean by this approximation can be explained as follows. It is well known that perturbations do not remain a soliton submanifold invariant. It means that in general the first-order multisoliton perturbation theory, besides the deformation of N-soliton dynamics, includes radiational effects. It is because perturbations mix discrete and continuous parts of a spectrum of a Lax operator [11] of a given soliton system. The adiabatic approximation neglecting radiational effects is restricted only to the soliton's deformation under various perturbations. Although such approximation weakens the power of the theory, nevertheless, in the one soliton case, it has been the most popular perturbation approach with well-recognized limitations. Here, we extend this approximation to the multisoliton case. The time evolution of H_n and T_n along the perturbed flow (4.6) are as follows:

$$\frac{\mathrm{d}H_n}{\mathrm{d}t} = L_{K_m+\varepsilon R} H_n = \{H_m, H_n\}_{\theta_0} + \varepsilon\langle \nabla H_n, R\rangle = \varepsilon\langle \nabla H_n, R\rangle \tag{4.7a}$$

$$\begin{aligned}\frac{\mathrm{d}T_n}{\mathrm{d}t} &= L_{K_m+\varepsilon R} T_n = \{H_m, T_n\}_{\theta_0} + \varepsilon\langle \nabla T_n, R\rangle \\ &= -(n+m+\alpha)H_{n+m} + \varepsilon\langle \nabla T_n, R\rangle.\end{aligned} \tag{4.7b}$$

From the explicit form (4.3a) of the conserved quantity H_n we find

$$\frac{\mathrm{d}H_n}{\mathrm{d}t} = \frac{\beta}{n+\alpha}\sum_{i=1}^{N} (c_i^{n+\alpha})_t = \beta \sum_{i=1}^{N} c_i^{n-1+\alpha}(c_i)_t. \tag{4.8}$$

Hence, substituting (4.8) and the explicit form (4.4a) of ∇H_n to the equation (4.7a), we obtain

$$\sum_{i=1}^{N} c_i^{n-1}\left(\beta c_i^{\alpha}(c_i)_t - \varepsilon\int_{-\infty}^{+\infty} A_i^* R(u_N)\,\mathrm{d}x\right) = 0. \tag{4.9}$$

Finally, from the arbitrariness of n, we find the time dependence of soliton velocities:

$$(c_i)_t = \frac{\varepsilon}{\beta} c_i^{-\alpha}\int_{-\infty}^{+\infty} A_i^* R(u_N)\,\mathrm{d}x. \tag{4.10}$$

Analogously, from the explicit form (4.3b) of the conserved quantities T_n, we have

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} = -\beta\sum_{i=1}^{N}[(n+\alpha)q_i c_i^{n-1+\alpha}(c_i)_t + c_i^{n+\alpha}(q_i)_t]. \tag{4.11}$$

Substituting (4.11), the explicit form (4.4b) of ∇T_n and the time dependence (4.10) of c_i to (4.7b), we obtain

$$\sum_{i=1}^{N} c_i^{n+\alpha}\left(-\beta(q_i)_t + \beta c_i^{m} - \varepsilon c_i^{-\alpha}\int_{-\infty}^{+\infty} B_i^* R(u_N)\,\mathrm{d}x\right) = 0 \tag{4.12}$$

and, from the arbitrariness of n, we have the second final formula

$$(q_i)_t = c_i^{m} - \frac{\varepsilon}{\beta} c_i^{-\alpha}\int_{-\infty}^{+\infty} B_i^* R(u_N)\,\mathrm{d}x. \tag{4.13}$$

representing the time evolution of N-soliton phases.

5. Examples

5.1. The Korteweg–de Vries equation

The celebrated Korteweg–de Vries equation (KdV) [9, 10] is given by

$$u_t = u_{xxx} + auu_x = K_1(u) \tag{5.1}$$

where u is an element of the Schwartz space of rapidly decreasing functions $S(R)$. The hierarchies of commuting symmetries

$$K_n(u) = \phi^n(u)K_0(u) = \left(D^2 + \frac{a}{3}DuD^{-1} + \frac{a}{3}u\right)^n u_x = (\theta_0 J)^n u_x$$
$$= \left[\left(D^3 + \frac{a}{3}Du + \frac{a}{3}uD\right)D^{-1}\right]^n u_x \tag{5.2}$$

and mastersymmetries

$$\tau_n(u) = \phi^n(u)\tau_0(u) = \left(D^2 + \frac{a}{3}DuD^{-1} + \frac{a}{3}u\right)^n \left(\frac{x}{2}u_x + u\right) \tag{5.3}$$

fulfil the commutator relations [2]

$$[K_n, K_m] = 0 \qquad [\tau_n, K_m] = (m + \tfrac{1}{2})K_{n+m} \qquad [\tau_n, \tau_m] = (m-n)\tau_{n+m}. \tag{5.4}$$

Here D denotes the differential operator WRT the x variable and D^{-1} its inverse:

$$D = \frac{\partial}{\partial x} \qquad D^{-1} = \int_{-\infty}^{x} \ldots \mathrm{d}\xi. \tag{5.5}$$

The scaling mastersymmetry is a Hamiltonian vector field WRT the second implectic structure

$$S(u) = \tau_0(u) = \left(D^3 + \frac{a}{3}Du + \frac{a}{3}uD\right) \operatorname{grad} \frac{3}{2a}\int_{-\infty}^{+\infty} xu \, \mathrm{d}x = \theta_0 \nabla F. \tag{5.6}$$

The N-soliton solutions decomposing at $\pm\infty$ into one-soliton solutions are given by [11]

$$u_N \cong \sum_{i=1}^{N} \frac{3}{a} c_i \operatorname{sech}^2[\tfrac{1}{2}\sqrt{c_i}\,(x + c_i t + q_i)] \tag{5.7}$$

where c_i are the eigenvalues of the recursion operator ϕ. Since $\alpha = \frac{1}{2}$ theorem 1 gives the action/angle variables WRT θ_0:

$$\sqrt{E_i} = -\frac{\partial F}{\partial q_i} = -\frac{3}{2a}\int_{-\infty}^{+\infty} xu_{q_i} \, \mathrm{d}x \tag{5.8a}$$

$$\Omega_i = -\frac{\partial F}{\partial \sqrt{c_i}} = -\frac{3}{a}\sqrt{c_i}\int_{-\infty}^{+\infty} xu_{c_i} \, \mathrm{d}x. \tag{5.8b}$$

The soliton non-canonical action/angle vector fields and corresponding scalar fields are of the form

$$K_n(u_N) = \phi^n \sum_{i=1}^{N} u_{q_i} = \sum_{i=1}^{N} c_i^n u_{q_i} \tag{5.9a}$$

$$S_n^p(u_N) = \phi^n \sum_{i=1}^{N} (c_i u_{c_i} - (n-p+\tfrac{1}{2}) q_i u_{q_i}) \tag{5.10a}$$

$$H_n(u_N) = -\frac{3}{a}\frac{1}{2n+1}\int_{-\infty}^{+\infty} x K_n(u_N)\,\mathrm{d}x = \frac{2}{2n+1}\sum_{i=1}^{N} c_i^n \sqrt{E_i} \tag{5.9b}$$

$$T_n(u_N) = -\frac{3}{a}\frac{1}{2n}\int_{-\infty}^{+\infty} x S_n^0(u_N)\,\mathrm{d}x = -\sum_{i=1}^{N} c_i^{n+1/2}\Omega_i. \tag{5.10b}$$

Additionally, the soliton non-Hamiltonian mastersymmetries (5.3) are as follows:

$$\tau_n(u_N) = \phi^n \sum_{i=1}^{N} (c_i u_{c_i} - \tfrac{1}{2} q_i u_{q_i}). \tag{5.11}$$

For the Lie algebra isomorphism P we easily calculate with $\beta = 18/a^2$

$$P(\sqrt{E_i}) = -\frac{3}{2a} P\left(\int_{-\infty}^{+\infty} x u_{q_i}\,\mathrm{d}x\right) = \beta\sqrt{c_i} \tag{5.12a}$$

$$P(\Omega_i) = -\frac{3}{a}\sqrt{c_i}\, P\left(\int_{-\infty}^{+\infty} x u_{c_i}\,\mathrm{d}x\right) = \beta q_i \tag{5.12b}$$

$$P(H_n) = \frac{\beta}{n+\frac{1}{2}}\sum_{i=1}^{N} c_i^{n+1/2} = \beta \bar{H}_n \tag{5.12c}$$

$$P(T_n) = -\beta \sum_{i=1}^{N} c_i^{n+1/2} q_i = \beta \bar{T}_n. \tag{5.12d}$$

One should notice that on M_N

$$\sum_{i=1}^{N} u_{q_i} = u_x \qquad \sum_{i=1}^{N} (c_i u_{c_i} - \tfrac{1}{2} q_i u_{q_i}) = u + \tfrac{1}{2} x u_x \tag{5.13}$$

holds, hence $K_n(u_N)$ and $\tau_n(u_N)$ are expressible by u_N and its x-derivatives (integrals) contrary to $S_n^p(u_N)$ which additionally contain derivatives of field variable u_N with respect to the asymptotic data.

Now we may pass to the N-soliton perturbation theory. As $\theta_0 = D^3 + \frac{1}{3}a(Du + uD)$ and $J = D^{-1}$, then

$$A_i^* = D^{-1}\frac{\partial u_N}{\partial q_i} = u^{(i)} \tag{5.14a}$$

$$B_i^* = D^{-1}\frac{\partial u_N}{\partial c_i} \tag{5.14b}$$

where $u^{(i)}$ are the so-called interacting solitons [12, 13]. Expressing the ith velocity c_i by the spectral parameter $\varkappa_i$ as $c_i = 4\varkappa_i^2$, we find the perturbed N-solution in the form

$u_N(\varkappa_1(t), \ldots \varkappa_N(t), q_1(t), \ldots, q_N(t))$, where

$$(\varkappa_i)_t = \varepsilon \frac{a^2}{18} \frac{1}{16\varkappa_i^2} \int_{-\infty}^{+\infty} R(u_N) u^{(i)} \, dx \tag{5.15a}$$

$$(q_i)_t = (2\varkappa_i)^{2m} - \varepsilon \frac{a^2}{18} \frac{1}{16\varkappa_i^2} \int_{-\infty}^{+\infty} R(u_N) D^{-1} \frac{\partial u_N}{\partial \varkappa_i} \, dx. \tag{5.15b}$$

In the particular case when $N = 1$ and $m = 1$, we find

$$u_s = \frac{12}{a} \varkappa^2 \operatorname{sech}^2 z \qquad z = \varkappa(x + q(t)) \tag{5.16a}$$

$$\varkappa_t = \frac{a}{6} \frac{\varepsilon}{4\varkappa} \int_{-\infty}^{+\infty} R(u_s) \operatorname{sech}^2 z \, dz \tag{5.16b}$$

$$q_t = 4\varkappa^2 - \frac{a}{6} \frac{\varepsilon}{4\varkappa^3} \int_{-\infty}^{+\infty} R(u_s)(z + \tfrac{1}{2} \sinh 2z) \operatorname{sech}^2 z \, dx \tag{5.16c}$$

which is the well-known result of one-soliton KdV perturbation theory [14].

5.2. *The Benjamin–Ono equation*

Our second example is the Benjamin–Ono (BO) equation [15, 16]

$$u_t = 4auu_x + \mathscr{H} u_{xx} = D \operatorname{grad} \int_{-\infty}^{+\infty} (\tfrac{1}{2} u \mathscr{H} u_x + \tfrac{2}{3} a u^3) \, dx \tag{5.17}$$

where $u \in S_p(R)$ [17] and $\mathscr{H}$ stands for the Hilbert transform

$$(\mathscr{H} f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - x} \, d\xi \qquad \text{(principal value)}. \tag{5.18}$$

Contrary to the KdV case, for the BO equation, instead of the hereditary algebra (2.19) we have the non-canonical action/angle algebra [4] $(K_n, S_n^0) = (K_n, S_n)$

$$[K_n, K_m] = 0 \qquad [S_n, K_m] = (n + m + 1) K_{n+m} \qquad [S_n, S_m] = (m - n) S_{n+m} \tag{5.19}$$

where the first few vector fields of each kind are as follows:

$$\begin{aligned} &K_0 = u_x \qquad K_1 = 4auu_x + \mathscr{H} u_{xx} \\ &K_2 = (\tfrac{16}{3} a^2 u^3 + 4a \mathscr{H} u u_x + 4au \mathscr{H} u_x - \tfrac{4}{3} u_{xx})_x \ldots \end{aligned} \tag{5.20a}$$

$$\begin{aligned} &S_{-1} = \frac{1}{4a} \qquad S_0 = xu_x + u \qquad S_1 = 2xK_1 + 4au^2 + 3 \mathscr{H} u_x \\ &S_2 = 3xK_2 + 16a^2 u^3 + 16au \mathscr{H} u_x - 4au_x \mathscr{H} u + 20a \mathscr{H} uu_x - 2u_{xx} \ldots . \end{aligned} \tag{5.20b}$$

The corresponding scalar fields $H_n(u)$ and $T_n(u)$, such that $K_n = D \nabla H_n$ and $S_n = D \nabla T_n$, as are follows:

$$H_n = -\frac{1}{4a} \frac{1}{n+1} \int_{-\infty}^{+\infty} x K_{n+1} \, dx \tag{5.21a}$$

$$T_n = -\frac{1}{4a} \frac{1}{n+2} \int_{-\infty}^{+\infty} x S_{n+1} \, dx. \tag{5.21b}$$

The scaling vector field is a Hamiltonian vector field WRT $\theta_0 = D$:

$$S(u) = S_0(u) = D \operatorname{grad} \int_{-\infty}^{+\infty} \tfrac{1}{2} x u^2 \, \mathrm{d}x = \theta_0 \operatorname{grad} F(u). \tag{5.22}$$

The N-soliton solutions decomposing at $t \to \pm\infty$ into one-soliton solutions are given by [17]

$$u_N \simeq \sum_{i=1}^{N} \frac{1}{a} \frac{c_i}{c_i^2(x + c_i t + q_i)^2 + 1} \tag{5.23}$$

where c_i are the eigenvalues of a recursion operator ϕ which exists in implicit form. Since $\alpha = 1$, theorem 1 gives the canonical action/angle variables WRT θ_0:

$$E_i = -\frac{\partial F}{\partial q_i} = -\int_{-\infty}^{+\infty} x u u_{q_i} \, \mathrm{d}x \qquad \Omega_i = -\frac{\partial F}{\partial c_i} = -\int_{-\infty}^{+\infty} x u u_{c_i} \, \mathrm{d}x \tag{5.24a}$$

and suitable Hamiltonian vector fields

$$A_i = \frac{\partial u_N}{\partial q_i} \qquad B_i = \frac{\partial u_N}{\partial c_i}. \tag{5.24b}$$

The soliton non-canonical action/angle vector fields (5.20) and the corresponding scalar fields (5.21) expressible by basic vector and scalar fields (5.24) are of the form

$$K_n(u_N) = \sum_{i=1}^{N} c_i^n u_{q_i} \qquad S_n(u_N) = \sum_{i=1}^{N} (c_i^{n+1} u_{c_i} - (n+1) q_i c_i^n u_{q_i}) \tag{5.25a}$$

$$H_n(u_N) = \frac{1}{n+1} \sum_{i=1}^{N} c_i^n E_i \qquad T_n(u_N) = -\sum_{i=1}^{N} c_i^{n+1} \Omega_i. \tag{5.25b}$$

Additionally, on M_N there exist non-Hamiltonian mastersymmetries

$$\tau_n(u_N) = \sum_{i=1}^{N} (c_i^{n+1} u_{c_i} - q_i u_{q_i}) \tag{5.26}$$

and $(K_n(u_N), \tau_n(u_N))$ constitute the hereditary algebra

$$\begin{gathered} [(K_n(u_N), K_m(u_N)] = 0 \qquad [\tau_n(u_N), K_m(u_N)] = (m+1) K_{n+m}(u_N) \\ [\tau_n(u_N), \tau_m(u_N)] = (m-n) \tau_{n+m}(u_N). \end{gathered} \tag{5.27}$$

For the map P (3.10) we find

$$P(E_i) = \beta c_i \qquad P(\Omega_i) = \beta q_i \qquad P(H_n) = \beta \bar{H}_n \qquad P(T_n) = \beta \bar{T}_n \tag{5.28}$$

where $\beta = \pi / 4a^2$.

Now let us consider the perturbed mth equation of BO hierarchy:

$$u_t = K_m(u) + \varepsilon R(u). \tag{5.29}$$

According to our general formulae (4.10) and (4.13), the time evolution of N-soliton parameters is as follows:

$$(c_i)_t = \frac{4a^2}{\pi} \varepsilon \int_{-\infty}^{+\infty} (D^{-1} u_{q_i}) R(u_N) \, \mathrm{d}x \tag{5.30a}$$

$$(q_i)_t = c_i^m - \frac{4a^2}{\pi} \varepsilon \int_{-\infty}^{+\infty} (D^{-1} u_{c_i}) R(u_N) \, \mathrm{d}x. \tag{5.30b}$$

In the particular case of one soliton we find

$$c_t = \frac{4a}{\pi}\varepsilon\int_{-\infty}^{+\infty}\frac{R(u_s)}{1+z^2}\,dz \qquad z = c(x+q(t)) \tag{5.31a}$$

$$q_t = c^m - \frac{4a}{\pi}\frac{\varepsilon}{c^2}\int_{-\infty}^{+\infty}\frac{zR(u_s)}{1+z^2}\,dz. \tag{5.31b}$$

5.3. Other examples of adiabatically perturbed multisolitons

First let us consider the perturbed modified Korteweg-de Vries (MKdV) equation

$$u_t = u_{xxx} + au^2u_x + \varepsilon R(u). \tag{5.32}$$

According to the results of I we have $J = D^{-1}$, $\alpha = \frac{1}{2}$, $\beta = 3/a$ and $c_i = 4\varkappa_i^2$, where $\varkappa_i$ is a spectral parameter. We find the adiabatically perturbed N-soliton solution in the form $u_N(\varkappa_1(t), \ldots q_N(t))$ where

$$(\varkappa_i)_t = \varepsilon\frac{a}{12}\int_{-\infty}^{\infty}R(u_N)D^{-1}\frac{\partial u_N}{\partial q_i}\,dx \tag{5.33a}$$

$$(q_i)_t = 4\varkappa_i^2 - \varepsilon\frac{a}{12}\int_{-\infty}^{\infty}R(u_N)D^{-1}\frac{\partial u_N}{\partial \varkappa_i}\,dx. \tag{5.33b}$$

One can find the explicit form of the N-soliton solution u_N in standard text books (see for example [13]). In a particular one-soliton case we find for $a = 6$

$$u_s = 2\varkappa\,\mathrm{sech}(z) \qquad z = 2\varkappa(x+q(t)) \qquad q(t) = 4\varkappa^2 t + q_0 \tag{5.34a}$$

$$\varkappa_t = \tfrac{1}{2}\varepsilon\int_{-\infty}^{\infty}\mathrm{sech}\,z\,R(u_s)\,dz \tag{5.34b}$$

$$q_t = 4\varkappa^2 - \frac{\varepsilon}{4\varkappa^2}\int_{-\infty}^{\infty}z\,\mathrm{sech}\,z\,R(u_s)\,dz \tag{5.34c}$$

which is the well-known result of the inverse scattering approach in the adiabatic one-soliton perturbation [18].

Second we consider the Caudrey-Dodd-Gibbon-Sawada-Kotera (CDGSK) equation with a perturbation term:

$$u_t = u_{5x} + \tfrac{5}{2}auu_{3x} + \tfrac{5}{2}au_xu_{2x} + \tfrac{5}{4}a^2u^2u_x + \varepsilon R(u). \tag{5.35}$$

For the above system $J = D^{-1}(D^2 + \frac{1}{2}au)D(D^2 + \frac{1}{2}au)D^{-1}$, $\alpha = \frac{1}{2}$, $\beta = 2/a^2$ and $c_i = (2\varkappa_i)^6$ hold, so the time evolution of the N-soliton parameters are of the following form:

$$(\varkappa_i)_t = \frac{a^2}{24}\varepsilon\int_{-\infty}^{\infty}R(u_N)J\frac{\partial u_N}{\partial q_i}\,dx \tag{5.36a}$$

$$(q_i)_t = (2\varkappa_i)^4 - \frac{a^2}{24}\varepsilon\int_{-\infty}^{\infty}R(u_N)J\frac{\partial u_N}{\partial \varkappa_i}\,dx. \tag{5.36b}$$

In the particularly interesting one-soliton case

$$u_s = \frac{12}{a}\varkappa^2\,\mathrm{sech}^2 z \qquad z = \varkappa(x+q(t)) \qquad q(t) = (2\varkappa)^4 t + q_0 \tag{5.37a}$$

the formulae (5.36) are confined to the form

$$\varkappa_t = \frac{a}{8\varkappa}\varepsilon \int_{-\infty}^{\infty} (\operatorname{sech}^2 z - \tfrac{3}{4}\operatorname{sech}^4 z) R(u_s)\,\mathrm{d}z \tag{5.37b}$$

$$q_t = (2\varkappa)^4 - \frac{a}{8\varkappa^3}\varepsilon \int_{-\infty}^{\infty} \operatorname{sech}^2 z(z + \tfrac{1}{4}\sinh 2z - \tfrac{3}{4}\tanh z - \tfrac{3}{4}z\operatorname{sech}^2 z) R(u_s)\,\mathrm{d}z. \tag{5.37c}$$

6. Summary

Two main results were obtained in this paper. First, for soliton systems (1.1)–(1.4), we proved the existence of a recursion operator (although sometimes in an implicit form), of the hierarchies of Hamiltonian and non-Hamiltonian mastersymmetries, and of a complete set of conservation laws (at least on the soliton submanifold). Moreover, we found these new symmetries and conservation laws in an explicit form. Second, we constructed the multisoliton perturbation theory in the so-called adiabatic approximation and found the explicit formulae of time dependence of multisoliton parameters. The results were illustrated by the KdV, the MKdV, the BO and the CDGSW equations. One should notice that all results were obtained in a purely algebraic way without using the inverse scattering method at all.

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